Possible physical meaning of the Tsallis entropy parameter

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Since the proposal of the Tsallis generalized entropy, the general explanation of the role played by the parameter q that defines which specific entropy to pick among a whole family, remained somewhat partial and tentative, although some particular examples were taken into account specifically. The purpose of the present paper is to present a rigorous formal derivation of a mathematical expression giving the parameter q in terms of the momentum fluctuations of a stochastic process, thus furnishing at least one of its possible physical origins.

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I. INTRODUCTION

In 1988 Tsallis proposed a generalized entropy mathematically given by [1]

$$S_q = k_B \frac{1-\sum_i p_i^q}{q-1},$$

where k_B is the Boltzmann constant, which for the equiprobability case, like in the microcanonical ensemble, becomes

$$S_q = k_B \frac{\rho^{1-q}-1}{1-q};$$

this expression in the limit $q \to 1$ reproduces the Boltzmann-Gibbs entropy

$$S_1 = -k_B \ln \rho.$$

This generalization brought about the problem of knowing the physical interpretation of the parameter q labeling the whole set of entropies. Other sets of entropies were also proposed [2], each one with its own set of properties, but we will focus only on the Tsallis generalization in this paper. I will be using the equiprobability case for reasons that will soon become clear.

Very recently I presented an axiomatic method allowing the derivation of the Schrödinger equation based upon three quite simple postulates [3] (hereafter Paper I). This derivation was then shown to be equivalent to another derivation, also made by me [4] (Paper II), based upon the notion of entropy; I showed then that the Schrödinger equation is mathematically a tributary of the concept of the Boltzmann-Gibbs entropy. In a further development of the derivation, I also proved [5] (Paper III) the derivations of Papers I and II to be mathematically equivalent to the stochastic derivation of de la Peña and Cetto [6] and a somewhat complete interpretation of the formalism in terms of fluctuations (stochasticity) was advanced.

The derivation method of the usual Schrödinger equation presented in Paper II was then easily extended by Olavo *et al* [7] (Paper IV) to embrace the Tsallis generalization of the Boltzmann-Gibbs entropy and a Schrödinger-type equation was found, which reduces to the usual linear Schrödinger equation in the $q \to 1$ limit, as required. We called this equation the generalized Schrödinger equation. In the derivation done in Paper IV (to which the reader is referred for the details) we wrote (for the equiprobability case)

$$\rho(x;t) = \left(1 + \frac{[1-q]}{k_B} S_q(x;t)\right)^{1/(1-q)} \tag{1}$$

and used the Liouville equation and the notion of fluctuation to find the generalized Schrödinger equation, given by

$$-\frac{\hbar_q^2}{2m}\left[\rho(x;t)^{1-q}\frac{\partial^2 \ln \psi_q(x;t)}{\partial x^2} + q\left\{\frac{\partial \ln \psi_q(x;t)}{\partial x}\right\}^2\right] + V(x) = i\hbar_q \sqrt{q}\,\frac{\partial \ln \psi_q(x;t)}{\partial t}, \tag{2}$$

where

$$\hbar_q = \sqrt{\frac{3-q}{2}}\,\hbar$$

and

$$\ln \psi_q(x;t) = \frac{1}{2k_B} S_q(x;t) + \frac{i s(x;t)}{\hbar_q \sqrt{q}} \tag{3}$$

with $s(x;t)$ related to the mean momentum of the system through $p(x,t) = \partial s(x,t)/\partial x$ and

$$\rho(x;t) = [1 + (1-q)\ln\{\psi_q^*(x;t)\psi_q(x;t)\}]^{1/(1-q)}. \tag{4}$$

The equation above, for reasons presented in Paper IV, is valid for the whole allowed interval $-\infty < q < 3$. It is important to stress that, when $q < 0$ we have a hyperbolic-type equation given by

$$-\frac{\hbar_q^2}{2m}\left[\rho(x;t)^{1-q}\frac{\partial^2 \ln \psi_q(x;t)}{\partial x^2} + q\left\{\frac{\partial \ln \psi_q(x;t)}{\partial x}\right\}^2\right] + V(x) = -\hbar_q \sqrt{|q|}\,\frac{\partial \ln \psi_q(x;t)}{\partial t},$$

and not the parabolic-type equation that is obtained for $q>0$. Equation (2) is the expression we must find by our stochastic derivation to be presented in the next section. Now it is possible to justify why we are using the equiprobability expression for the probability density; the derivation method allows us to find a generalized Schrödinger equation (or a Schrödinger equation, in the usual case), which is related, *for each probability density*, with the *same energy* (we could have labeled the densities by quantum numbers to make this point even more explicit). In the process of derivation (see Paper IV) we also found that the average of the momentum fluctuations taken over momentum space alone is given by

$$\overline{\delta p(x)^2}=-\frac{\hbar_q^2}{4k_B}\rho(x)^{q-1}\frac{\partial^2 S_q}{\partial x^2}, \tag{5}$$

where the result still depends upon x, in general, since the average process was taken only over momentum space.

In the next section we will extend this generalization to the developments made in Paper III and show what kind of stochastic process is related to this generalized Schrödinger equation. With these mathematical developments and also the extension of the interpretation already presented in Papers I–III, we will be capable to set forth at least one possible interpretation (in terms of fluctuations) for the role of the parameter q in actual physical systems. This is the aim of the present paper.

The paper is arranged in the following manner. In the second section we will show how we can generalize the stochastic derivation of Paper III to obtain again the generalized Schrödinger equation (2). In the third section we will show that the derivation of Paper III and the present stochastic one are fully equivalent (by means of the Onsager relations) and this will make it possible for us to show very clearly the role of the parameter q within the formalism. The present results have the advantage of being totally formal and mathematically exact, while furnishing quite a direct physical interpretation. In the fourth section we will present the interpretation of the results found in the previous sections and also reformulate the mathematical appearance of the theory to put it into a Newtonian format (which is consistent with the stochastic approach and also much easier to grasp). Section V will be devoted to an application of the formal developments of the previous sections. The last section will be reserved to our concluding remarks.

II. STOCHASTIC DERIVATION

The present stochastic derivation will follow very closely the general lines of the one made by de La Peña and Cetto [6] and will be just a generalization of their result.

Now we will consider $x(t)$ as a stochastic process. This means that the velocity related with this process cannot be obtained by its direct derivation, for $x(t)$ is not, in general, differentiable.

In this case we have to introduce a finite time interval Δt, small compared with the characteristic times of the systematic movement (the one related with Newton's equation), but large enough compared with the correlation time of the *fluctuating force* (the so-called coarse graining).

Since Δt is a very small time interval, we may write the following expansion:

$$\frac{1}{\Delta t}[f(x(t+\Delta t),t+\Delta t)-f(x(t),t)] \approx \frac{\partial f}{\partial t}+\frac{1}{\Delta t}\exp\left[\rho(x;t)^{1-q}\delta x\frac{\partial}{\partial x}\right]f, \tag{6}$$

where f is an arbitrary function; the difference from the present calculation to the one already found in the literature [6] is simply the extra factor $\rho(x;t)^{1-q}$ in the exponent of the translation operator. This means that, contrary to the usual stochastic process that furnishes the linear Schrödinger equation, translations within the system by δx will be dependent on the position where the translation takes place (more specifically they will depend upon the density at the point where the translations take place, which already explains the source of nonlinearity of the generalized Schrödinger equation). In the whole derivation process I might have written expression (6) as simply

$$\frac{\partial f}{\partial t}+\frac{1}{\Delta t}\exp\left[g(x;t)\delta x\frac{\partial}{\partial x}\right]f,$$

and then looked for the function $g(x;t)$ as an ansatz that will furnish the correct equation. In what follows I preferred to use the factor $\rho(x;t)^{1-q}$ directly, which is a restricted case of the general ansatz, since it makes the derivation easier to understand.

The above expression (6) may be rewritten as

$$\frac{1}{\Delta t}[f(x(t+\Delta t),t+\Delta t)-f(x(t),t)] \approx\left[\frac{\partial f}{\partial t}+\frac{1}{\Delta t}\delta x\rho(x;t)^{1-q}\frac{\partial f}{\partial x} +\frac{1}{2\Delta t}(\delta x)^2\rho(x;t)^{1-q}\frac{\partial}{\partial x}\left\{\rho(x;t)^{1-q}\frac{\partial f}{\partial x}\right\}\right]. \tag{7}$$

Using expression (7) and repeating without modifications all the developments made in Paper III (to which the reader is strongly referred to), we can find the following equations:

$$\frac{\partial v}{\partial t}+v\frac{\partial v}{\partial x}-\nu_-\frac{\partial}{\partial x}\left(\rho^{1-q}\frac{\partial v}{\partial x}\right)-\lambda u\frac{\partial u}{\partial x}-\lambda\nu_+\frac{\partial}{\partial x}\left(\rho^{1-q}\frac{\partial u}{\partial x}\right) =f_0/m \tag{8}$$

and

$$\frac{\partial u}{\partial t}+v\frac{\partial u}{\partial x}+u\frac{\partial v}{\partial x}+\nu_+\frac{\partial}{\partial x}\left(\rho^{1-q}\frac{\partial v}{\partial x}\right)-\nu_-\frac{\partial}{\partial x}\left(\rho^{1-q}\frac{\partial u}{\partial x}\right)=0, \tag{9}$$

where $u(x,t)$ is the stochastic velocity and $v(x,t)$ is the systematic velocity and v_- and v_+ are characteristic constants of the derivation (see Paper III) that will be fixed in what follows.

In the Newtonian limit, where there are no fluctuations, we have

$$\nu_+ = \nu_- = 0 \rightarrow u = 0 \tag{10}$$

thus giving

$$m\frac{dv}{dt} = f_0 = -\frac{\partial V(x)}{\partial x}, \tag{11}$$

as desired. Equation (8) and (9) are the main equations governing the dynamics of the system.

We now substitute in Eqs. (8) and (9) the expressions

$$\nu_- = 0; \quad mv = p(x;t) = \frac{\partial s(x;t)}{\partial x} \tag{12}$$

and write the stochastic velocity as

$$u = \frac{\hbar_q}{2mk_B}\frac{\partial S_q}{\partial x} \tag{13}$$

to find the two equations

$$\frac{\partial}{\partial x}\left[\frac{\partial s}{\partial t} + \frac{1}{2m}\left(\frac{\partial s}{\partial x}\right)^2 + V(x) - \frac{\lambda \hbar_q^2}{8m}\left(\frac{1}{k_B}\frac{\partial S_q}{\partial x}\right)^2 - \frac{\lambda \nu_+ \hbar_q}{2}\left(\frac{\rho^{1-q}}{k_B}\frac{\partial^2 S_q}{\partial x^2}\right)\right] = 0 \tag{14}$$

and

$$\frac{\partial}{\partial x}\left(\frac{\partial S_q}{\partial t} + \frac{1}{k_B}\frac{\partial S_q}{\partial x}\frac{1}{m}\frac{\partial s}{\partial x} + \frac{2\nu_+ \rho^{1-q}}{\hbar_q}\frac{\partial^2 s}{\partial x^2}\right) = 0. \tag{15}$$

These last two equations, with the substitution

$$\lambda = q,$$

$$\nu_+ = \hbar_q/(2qm) \tag{16}$$

give, finally,

$$\frac{\partial s}{\partial t} + \frac{1}{2m}\left(\frac{\partial s}{\partial x}\right)^2 + V(x) - \frac{\hbar_q^2}{4m}\left[\left(\frac{\rho^{1-q}}{k_B}\frac{\partial^2 S_q}{\partial x^2}\right) + \frac{q}{2}\left(\frac{1}{k_B}\frac{\partial S_q}{\partial x}\right)^2\right] = 0 \tag{17}$$

and

$$\frac{\partial S_q}{\partial t} + \frac{1}{k_B}\frac{\partial S_q}{\partial x}\frac{1}{m}\frac{\partial s}{\partial x} + \frac{\rho^{1-q}}{qm}\frac{\partial^2 s}{\partial x^2} = 0. \tag{18}$$

These two equations are exactly those we get if we substitute expression (3) into the generalized Schrödinger equation (2), meaning that they are equivalent, as we wanted to show.

III. CONNECTION BETWEEN THE DERIVATIONS

The results of the present section are a straightforward generalization of those derived by me in a previous paper [5]. However, in the present generalized formulation they will furnish precisely the result we are looking for.

We may begin by rewriting Eq. (13),

$$u = \frac{\hbar_q}{2mk_B}\frac{\partial S_q}{\partial x}, \tag{19}$$

and remember that, for our fluctuating system, we expect the linear Onsager relations to be applicable [7] (the Onsager relations were proved to be also valid within the Tsallis statistics, as shown in [8]),

$$u = \alpha\frac{\partial S_q}{\partial x}, \tag{20}$$

where α is the so-called "friction coefficient" given by the expression [9]

$$\alpha = \frac{1}{m^2 k_B}\int_{-\infty}^{0} E^{(q)}[\delta p(0)\delta p(s)]_0 ds = \frac{1}{m^2 k_B}\int_{-\infty}^{0} ds\int_{-\infty}^{+\infty}\rho(x)^q[\delta p(x,0)\delta p(x,s)]dx, \tag{21}$$

where $E^{(q)}(f)$ means the expectation value of the argument f taken using the Tsallis procedure. Equations (19), (20), and (21) give

$$\hbar_q = \hbar\sqrt{\frac{3-q}{2}} = \frac{2}{m}\int_{-\infty}^{0} ds\int_{-\infty}^{+\infty}\rho(x)^q[\delta p(x,0)\delta p(x,s)]dx; \tag{22}$$

but, since $\hbar$ is a constant, which may be fixed by experiment using the common value obtained when $q=1$, the expression above may be considered as the very definition of the parameter q and it unequivocally shows the connection between this parameter and the average momentum fluctuations. This is precisely the result we were willing to obtain and it furnishes a physical connection between q and the stochastic processes taking place within the considered subsystem. We note, however, that the parameter q appears at both sides of expression (22); this is so because the averages have to be calculated using the probability function that underlies the choice of the entropy. Thus, expression (22) furnishes, in principle and theoretically, a transcendental equation giving the value of q, whenever one is able to calculate the integral on the right-hand side of expression (22). However, from the experimental point of view, if one is able to determine experimentally the value of the "friction coefficient" related with some stochastic process, its departure from the value of the usual Planck's constant will furnish, experimentally, the underlying value of the parameter q.

IV. FURTHER RESULTS

We have said in the last section that Eqs. (17) and (18) give the whole dynamical behavior of the system. It is quite interesting to write these equations in a somewhat different fashion that may enlighten our understanding of the present subject. Indeed, it is very simple to show that Eq. (17) may be rewritten as

$$\frac{dp(x;t)}{dt}=-\frac{\partial}{\partial x}\left\{V(x)-\frac{\hbar_q^2}{4m}\left[\frac{\rho(x;t)^{1-q}}{k_B}\frac{\partial^2 S_q}{\partial x^2}+\frac{q}{2}\left(\frac{1}{k_B}\frac{\partial S_q}{\partial x}\right)^2\right]\right\}, \tag{23}$$

which is a Newton-like equation for an *average momentum* and an effective potential

$$V_{eff}(x)=V(x)-\frac{\hbar_q^2}{4m}\left[\frac{\rho(x;t)^{1-q}}{k_B}\frac{\partial^2 S_q}{\partial x^2}+\frac{q}{2}\left(\frac{1}{k_B}\frac{\partial S_q}{\partial x}\right)^2\right]. \tag{24}$$

The second term on the right-hand side of the previous expression gives the average alteration of the potential function by the momentum fluctuations (see Papers II and IV).

Thus, Eq. (23) furnishes the connection between the average dynamical behavior of the system and the statistics governing the behavior of the fluctuations. The second term on the right-hand side of expression (24) is nothing but a generalized version of Bohm's (so-called) quantum potential, whose origin (as we saw in Paper IV) is kinetic and comes from the fluctuations.

V. APPLICATION

We will now present a simple example of the above-mentioned results by solving analytically the harmonic oscillator problem for the ground state (the excited states are much more difficult to solve). Thus, the potential function is given by

$$V(x)=\frac{1}{2}m\omega^2x^2,$$

where m is the mass of the particle, ω is the frequency of the movement; the generalized Schrödinger equation (2) becomes

$$-\frac{\hbar_q^2}{2m}\left[\rho(x;t)^{1-q}\frac{\partial^2 \ln\psi_q(x;t)}{\partial x^2}+q\left\{\frac{\partial \ln\psi_q(x;t)}{\partial x}\right\}^2\right]+\frac{1}{2}m\omega^2x^2=i\hbar_q\sqrt{q}\,\frac{\partial \ln\psi_q(x;t)}{\partial t}$$

and has, as its solutions (the reader may verify this by direct substitution), the following.

(1) For $1<q<3$,

$$\psi_q(x;t)=e^C\exp\left(-\frac{m\omega}{2\hbar_q}x^2\right)\exp\left(-\frac{iE_q}{\sqrt{q}}t\right) \tag{25}$$

$$=e^C\exp\left(-\frac{m\omega}{2\hbar_q}x^2\right)\exp\left\{-\frac{i\omega}{2\sqrt{q}}[1-2(q-1)C]t\right\}, \tag{26}$$

with density given by [using Eq. (1)]

$$\rho(x,t)=\left\{[1+2(q-1)C]+\frac{(q-1)m\omega}{\hbar_q}x^2\right\}^{1/(1-q)} \tag{27}$$

and the constant C, given by the normalization integral (note the integration limits; there is no cutoff and the sample space is $[-\infty,+\infty]$)

$$\int_{-\infty}^{+\infty}\rho(x,t)^q dx=1,$$

being written as

$$C=\frac{1}{2(q-1)}\left\{1-\left[\sqrt{\frac{\hbar}{m\omega}}\left(\frac{3-q}{2}\right)^{1/4}\frac{\sqrt{\pi}\Gamma\left\{\frac{q+1}{2(q-1)}\right\}}{\sqrt{q-1}\Gamma\left(\frac{q}{q-1}\right)}\right]^{2(q-1)/(q+1)}\right\}. \tag{28}$$

The energy E_q becomes

$$E_q=[1+2(1-q)C]\frac{\hbar_q\omega}{2}. \tag{29}$$

(2) For $0<q<1$,

$$\psi_q(x;t)=e^C\exp\left(-\frac{m\omega}{2\hbar_q}x^2\right)\exp\left\{-\frac{i\omega}{2\sqrt{q}}[1+2(1-q)C]t\right\}$$

with the probability density given by

$$\rho(x;t)=\left\{[1-2(1-q)C]-\frac{(1-q)m\omega}{\hbar_q}x^2\right\}^{1/(1-q)},$$

where, now, the constant C, given by the integration (note that we have a cutoff, which is given by the positive and negative roots of the probability density, and the sample space must be finite)

$$\int_{-\alpha}^{+\alpha}\rho(x;t)^q dx=1,$$

becomes

$$C=\frac{1}{2(1-q)}\left\{\left[\sqrt{\frac{\hbar}{m\omega}}\left(\frac{3-q}{2}\right)^{1/4}\frac{\sqrt{1-q}\,\Gamma\left\{\frac{3-q}{2(1-q)}\right\}}{\sqrt{\pi}\,\Gamma\left(\frac{1}{1-q}\right)}\right]^{2(1-q)/(1+q)}-1\right\}.$$

The energy is, again,

$$E_q=[1+2(1-q)C]\frac{\hbar_q\omega}{2}.$$

This result justifies what we have said above about the use of the microcanonical ensemble; indeed, for each q we will be working with the states related with the same energy E_q.

(2) For $-\infty<q<0$,

$$\psi_q(x;t)=\exp\left[-\frac{1}{2(1-q)}\right]\exp\left(-\frac{m\omega}{2\hbar_q}x^2\right)\times\exp\left\{c\exp\left[-\frac{(1-q)}{\sqrt{|q|}}\omega t\right]\right\},$$

where c is a constant that could be obtained by fixing the normalization of the probability density at $t=0$. The probability density is given by (note that the cutoff now is a function of time and goes to zero as time passes, meaning that the sample space shrinks as time goes on)

$$\rho(x;t)=\left\{c\exp\left[-\frac{(1-q)}{\sqrt{|q|}}\omega t\right]-\frac{(1-q)m\omega}{\hbar_q}x^2\right\}^{1/(1-q)},$$

from which we note that we must have $c>0$. The energy, now, is a function of time, as expected. In Fig. 1 we plot the behavior of the energy E_q as a function of the parameter q.

With these solutions we may calculate the averages in Eq. (22) once we know the behavior of the fluctuations in the momenta. Just as an example of the method, suppose that we have found (by experimental means, for instance) that these fluctuations behave as

$$E^{(q)}[\delta p(x,0)\delta p(x,s)]=E^{(q)}\overline{[\delta p(x)]^2}\exp(-\gamma s),$$

meaning that the correlation of the two momentum fluctuations decays exponentially in time. Now we can use expression (5) to write Eq. (22) as

$$\hbar_q=\frac{2}{m}\int_{-\infty}^{0}\exp(-\gamma s)ds\int_{-\infty}^{+\infty}\rho(x)^q\overline{[\delta p(x)]^2}dx$$
$$=-\frac{\hbar_q^2}{2\gamma m k_B}\int_{-\infty}^{+\infty}\rho(x)^{2q-1}\frac{\partial^2 S_q}{\partial x^2}dx,$$

thus giving

$$-\frac{2\gamma m k_B}{\hbar_q}=\int_{-\infty}^{+\infty}\rho(x)^{2q-1}\frac{\partial^2 S_q}{\partial x^2}dx. \qquad (30)$$

Let us apply the above result to the $1<q<3$ case (where the sample space is $[-\infty,+\infty]$). We then have

$$\frac{\partial^2 S_q}{\partial x^2}=-\frac{2m\omega k_B}{\hbar_q}$$

and Eq. (30) becomes the transcendental equation

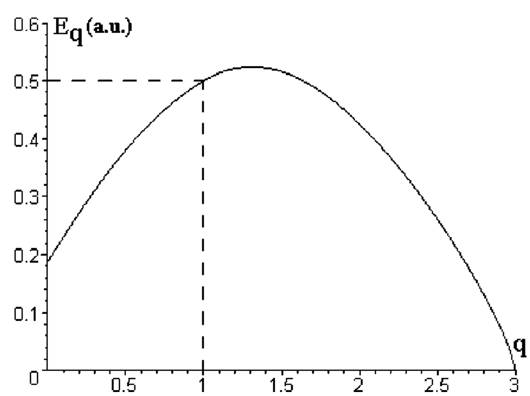

FIG. 1. The behavior of the ground state energy E_q of the harmonic oscillator with respect to the Tsallis statistical parameter q. Its value, when $q=1$ is $E_1=0.5$ a.u. as usual (we have made $\hbar=\omega=1$).

$$\frac{\gamma}{\omega}=\int_{-\infty}^{+\infty}\left\{[1+2(1-q)C]-\frac{m\omega}{\hbar_q}x^2\right\}^{(2q-1)/(1-q)}dx,$$

where C is given by Eq. (28). We can write the integral as

$$\beta^{\nu}\int_{-\infty}^{+\infty}\{1+\gamma x^2\}^{\nu}dx,$$

where $\beta=[1+2(1-q)C]$, $\gamma=(q-1)m\omega/\beta\hbar_q$, and $\nu=(2q-1)/(1-q)$. This integral may be easily calculated and gives

$$\beta^{\nu+1/2}\frac{\sqrt{\pi}\Gamma(-\nu-1/2)}{\sqrt{q-1}\Gamma(-\nu)}\sqrt{\frac{\hbar_q}{m\omega}},$$

and the transcendental equation becomes

$$\frac{\gamma}{\omega}=[1+2(1-q)C]^{-(3q-1)/[2(q-1)]}\frac{\sqrt{\pi}\Gamma\left[\frac{3q-1}{2(q-1)}\right]}{\sqrt{q-1}\Gamma\left(\frac{2q-1}{q-1}\right)}\left(\frac{3-q}{2}\right)^{1/4}\sqrt{\frac{\hbar}{m\omega}},$$

with

$$C=\frac{1}{2(q-1)}\left\{1-\left[\sqrt{\frac{\hbar}{m\omega}}\left(\frac{3-q}{2}\right)^{1/4}\frac{\sqrt{\pi}\Gamma\left\{\frac{q+1}{2(q-1)}\right\}}{\sqrt{q-1}\Gamma\left(\frac{q}{q-1}\right)}\right]^{[2(q-1)]/(q+1)}\right\}.$$

If we put $\hbar=\omega=1$, then we finally get

$$\gamma=\left[\left(\frac{3-q}{2}\right)^{1/4}\frac{\sqrt{\pi}\Gamma\left\{\frac{q+1}{2(q-1)}\right\}}{\sqrt{q-1}\Gamma\left(\frac{q}{q-1}\right)}\right]^{-[2(3q-1)]/(q+1)}\times\frac{\sqrt{\pi}\Gamma\left[\frac{3q-1}{2(q-1)}\right]}{\sqrt{q-1}\Gamma\left(\frac{2q-1}{q-1}\right)}\left(\frac{3-q}{2}\right)^{1/4},$$

which is a transcendental equation for q, if we know the value of γ, as assumed here. In Fig. 2 we plot the behavior of the right-hand side of this equation. We thus see that, for each value of γ (greater than approximately 0.3) there will be at least one value of q, that is, one choice for the Tsallis parameter giving the statistical behavior of the system. The calculations of the other two cases, related to the two other allowed intervals of q, are similar and will not be done here. In the next section we will present a deeper interpretation of these results.

VI. CONCLUSIONS

The physics governing the interpretation of Eq. (23), for the particular case $q=1$, was already developed in our previous papers [3–5]. It is opportune, however, to develop its generalized version. In the present framework we are considering one single system where a force field [with a physical potential function $V(x)$] is responsible for the interaction of the particles composing the system. This system is a closed one, since no other external force field is present in the exact Newtonian equations governing the movement of each particle. The total energy of the system is, therefore, conserved. Now we make the decision of treating the closed single system as composed by two subsystems: the particles (eventually, only one) and the force field, each one capable of keeping some amount of energy. We also choose to describe the parameters of the subsystem composed by the particles, while abstracting from those related with the force field, which then turns into the thermal reservoir of the whole system. With the adoption of this strategy of description, statistical physics tells us that there will appear fluctuations re-

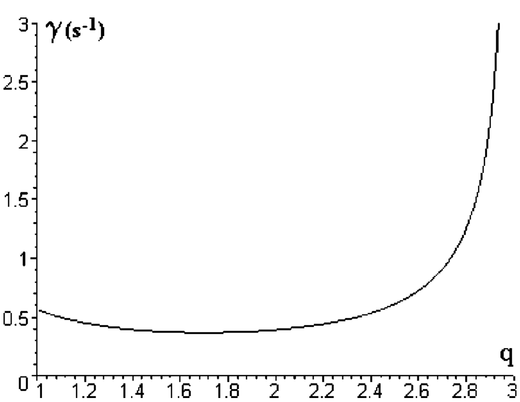

FIG. 2. The behavior of the parameter γ with respect to the Tsallis statistical parameter q for the ground state of the harmonic oscillator. Note the asymptotic behavior near $q=3$ (we have made $\hbar=\omega=1$).

sponsible for the exchange of energy between the two subsystems. In this case, for instance, the energy of the particles, being taken into account explicitly, will fluctuate, sometimes being lowered by transferring energy to the force field, sometimes being increased taking energy from the force field. This means that the average potential, governing the average movement of the particles will not be given simply by $V(x)$, but we will have to correct it to take into account, as an average, the energy fluctuations. This correction is precisely the one given by the second term on the right hand side of Eq. (24). Up to this point, the analysis is completely equivalent to the one we have made in Paper III [5]. In that paper, however, we have *assumed as an axiom* that the statistics governing the fluctuations is the one related with the Boltzmann-Gibbs entropy function. As far as generalized entropies are considered, this may be understood as a mere wild guess (very fruitful, indeed, but still a guess). It is possible that, besides the Boltzmann-Gibbs entropy rule governing the behavior of the fluctuations (the energy exchange between the particle subsystem and the force field reservoir), there are other rules, depending upon some characteristics of the system that go beyond this latter entropy; some of them may well be modeled by the Tsallis generalization of the Boltzmann-Gibbs entropy rule for $q \neq 1$.

With respect to results in the range $q<0$, one may argue that, since the system of particles and field (reservoir) is closed there is no room for a dissipative solution; we stress that this is not the case here. What is being dissipated is the energy of the particle system by increasing the energy of the field system. Indeed, we can envisage such a situation when a particle makes transitions between two levels of energy, giving up a photon (which is the same as giving energy to the electromagnetic field, for instance)—it is needless to say that this is a nonequilibrium situation, where we expect the range $q<0$ to become important.

We thus know that, given the type of fluctuations (known by any other method) or the energy of the system of particles, it is possible to discover, using expression (29) and/or Eq. (22), the appropriate value of the parameter q. In any case, expression (22) gives us the means to interpret the role of the parameter q.

There *is*, however, an advantage of working with the formalism related with Eq. (23). It allows us to visualize the average dynamical behavior of the system *as a function of the parameter* q. This may be of invaluable help in the investigation of the relations between this parameter and dynamical systems and may also help, we hope, finding the explicit connection between the parameter q and fractal behavior. However, we leave this analysis for a future paper.

There are a number of other alternatives for the Tsallis entropy generalization. They might be taken into account in the same fashion we did in the present paper. This, however, would be a mere mathematical exercise that, most probably, would not bring about any new fundamental explanation, and this is why we kept ourselves within the scope of the Tsallis generalization.

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